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ABSTRACT

This teacher's guide contains nine topics in number theory. Suggested questions for the teacher, short investigations, and possible exercises for the student are included. Chapter 1 is an introduction to sequences and series using geoboard activities involving triangular numbers, square numbers, rectangular numbers, and pentagonal numbers. The second chapter concerns prime numbers and generating prime numbers; included is a computer program in BASIC to find all prime numbers less than any given number. Chapter 3 concerns divisors, with geometrical interpretation of the greatest common factor. The next chapter is on congruence of numbers in modular arithmetic with exercises leading to divisibility tests. A team game with rules which can be used for reviewing number theory is presented followed by a chapter on Pythagorean systems. The final chapters of the guide present a brief description of linear diophantine equations. (JW)

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# TOPICS IN NUMBER THEORY
# THE NUMBER GAME

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TOPICS IN NUMBER THEORY

THE NUMBER GAME

Laj Batra (1969-70)

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# INTRODUCTION

This is a collection of select topics arranged to create an interesting unit in number theory.

This is by no means a complete introduction to the theory of numbers, and the teacher is encouraged to supplement with other materials.

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# I. Figurate Numbers: Number Patterns

This section is meant to be an introduction to sequences and series. Many important facts, principles and formulas can be derived by the student when given the opportunity to manipulate materials to see what is going on. This affords many opportunities for total student involvement in constructing geometric figures (plane) with pegs and peg board, examining the relationships which exist among them, and gaining insights about these relationships.

## Triangular Numbers

Using the peg board, have the students show the smallest simple plane geometric figure (using the least possible pegs).

1. What figure do you have?
2. How many pegs did you use?
3. Have students move to an adjacent column.
4. What is the least number of additional pegs you need to have one more peg in that column than in the previous one?
5. Use rubber bands to surround the pegs. What figure do you have?

6. Move to the next column and form the next larger triangle similar to the one before. How many pegs did you need to use here?
7. What is the relationship of the number of pegs in this column to the number in the previous one?

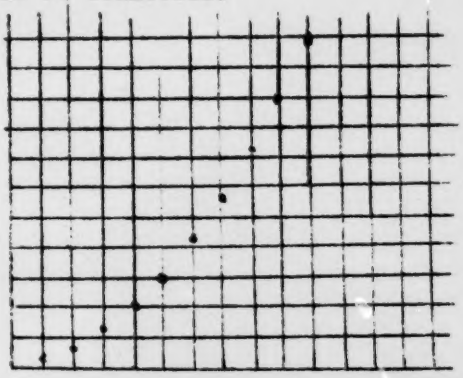
Repeat steps 3 - 7.

8. Can you predict the number of pegs you will need to use in the next column to get the next triangular number?
9. Can you state a formula to describe what we have been doing?
10. Use the formula to find out how many pegs will be in the nth row. Why?
11. What kind of progression do you have when you look at the numbers of pegs in each successive column?

The numbers (of pegs) in the successive columns form a special type of sequence of numbers.

12. What is the sum of the numbers in each column and the numbers of all the columns prior to it?
13. Is there a last triangular number? Why?

Using graph paper have the students draw the coordinates x and y, counting off the units on each axis. Using the formula derived above, $\left(\frac{n(n+1)}{2}\right)$ they should get the values of x and y for the first 10 numbers. These values are plotted on the graph by putting a dot in the appropriate intersection for each value of x and y. The plotted dots should be connected.



1. What does the curve look like?
2. Use the graph to predict what the thirteenth triangular number will be. The twenty-first.
3. List all the observations you can make about this sequence of numbers and the series generated from it.

Square Numbers

Using the peg board, have the students show the smallest simple plane geometric figure (using the least possible pegs).

1. Construct the first triangular number. How many pegs did you use?
2. Have the students construct the next triangular numbers.
3. How many more pegs will you need to construct a square from this triangle?
4. Leaving each figure intact, construct the next two triangular numbers See how many more pegs you will need to construct a square from each triangle.
5. What is the formula for deriving a square number? What is the 15th square number? the 25th?
6. What can you say about the diagonals of each triangle in relation to the square formed from it.
7. Can you derive some formulas for a triangular number from the formula for a square number?
8. Keeping the triangular numbers in mind, examine the following:

$0 + 1 = 1$

$1 + 3 = 4$

$3 + 6 = 9$

$6 + 10 = 16$

$10 + 15 = 25$

9. What can you say about the relationship expressed here?

10. Can you write some general expression for a square number from the above examples?

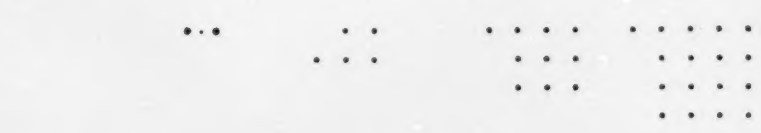
(The sum of two consecutive triangular numbers is a square number.)

11. Is $2(1/2\ n(n+1))$ the same as $1/2\ n\ (n + 1) + 1/2\ n\ (n + 1)$?

12. Graph each one using numbers from the set of triangular numbers. There may be other general expressions which should be tested.

When you wish to construct some $(n^2 + 1)$ graphs, you would go about it as follows:

(1) (2) (3) (4)



Exercises

1. Show the $(n^2 + 1)$ graphs for 5, 7 and 9.
2. Look at this graph:

   . . . . .
   . . . . .
   . . . . . .

   What is the graph in terms of n?
3. Show a $(n + 1)^2$ graph.
4. Compare the number of dots in the graph shown above and an $(n + 1)^2$ graph.

Rectangular Numbers

Using the peg board, have the students construct the smallest non-degenerate line figure.

1. What figure is this? . .
2. How many pegs did you use?
3. Using the least number of pegs, construct a non-degenerate rectangle (not a square) which includes the previous pegs. How many pegs did you use?
4. Using the least number of pegs, construct a rectangle so that each dimension is increased. How many pegs did you use?

5. List the successive numbers (of pegs) added so far.
6. List the successive sums of these numbers.
7. Repeat steps 4, 5 and 6 until you can answer the following questions.
    a) What are the rectangular numbers you have discovered?
    b) What is the next rectangular number? the 25th? the nth?

Exercises

1. Two successive rectangular numbers are called twin rectangular numbers. Can you determine the relationship between the sum of twin rectangular numbers and square numbers?
2. What cay you say about the differences of twin rectangular numbers?
3. What cay you say about the differences of consecutive differences?
4. What type of sequence is formed by the set of differences of twin rectangular numbers?
5. Can you use arithmetic progression in the determination of rectangular numbers?
6. Can you define twin triangular numbers so they may be used to determine rectangular numbers?

Pentagonal Numbers

Reproduce the following patterns and distribute to the class.

1 5 12 22

1. The set of numbers 1, 5, 12, 22 ... are called pentagonal numbers. Can you express each pentagonal number as the sum of a square and another number?
2. What can you say about the other number in each case?
3. What is the next pentagonal number after 22?
4. What is the formula for getting the nth pentagonal number?
5. Examine the following set of equations:

   $1 = 1$

   $1 + 4 = 5$

   $1 + 4 + 7 = 12$

   $1 + 4 + 7 + 10 = 22$

   What can you say about these equations?
6. Using the above set of equations, tell how you would get the next pentagonal number? the nth pentagonal number?

7. Can you derive several formulas for deriving pentagonal numbers? Check each one for the first four pentagonal numbers.

8. Are the pentagonal numbers for the negative numbers (-1, -2, ...,) the same as for the positive numbers (1, 2, 3, ...,)?

9. Prove that 3 times any triangular number plus 1 is a square number.

Short Investigation

Using the form of generalization A,

1. Write 12 and 18 as products of primes.

   Let S be the set of prime factors of 12, which consists of 2, 2, 3. Let T be the set of prime factors of 18 which consists of 3, 3, 2.

   a. Draw the Venn diagrams of S and T.

   b. Find $S \cap T$.

   c. Give product of the elements of $S \cap T$.

   d. Is this the same as the greatest common factor (divisor) found in question 2 of the previous Short Investigation?

You have just developed the greatest common factor.

2. Find the prime factors of each number in the pairs (7, 11) and (4, 13).

3. What is the greatest common divisor of each of the above pairs?

4. Can you make general statements at this time? If not, find the greatest common divisor for (37, 23), (36, 23). Make a generalized statement.

Definition: If a and b have a greatest common divisor of 1, they are known as relatively prime.

## II. Prime Numbers

Two definitions of prime numbers are given here. The first is given because it is commonly found in the literature and the second because it leads to some interesting discussion.

Definition 1: A prime is a non-zero whole number greater than 1, whose only divisors are itself and 1.

Definition 2: A prime is a non-zero whole number having exactly two divisors.

Short Investigation

(a) Keeping this in mind, find all the prime numbers between 1 and 50.

(b) What prime numbers are of the form $n^2 + 1$ where n is a positive number.

(c) How many even prime numbers are there in your list?

(d) How many primes numbers are there of the form $n^2 - 1$ where n is a positive whole number?

(e) In your list how many prime numbers are there of the form $q = 2p + 1$ where p is a prime?

(f) Write 6 as many ways as you can as the difference of two primes.

(g) Write 10 as many ways as you can as the difference of two primes.

(h) How many ways can you write 5, 7 and 9 as the difference of two primes?

(i) Make a generalized statement from (f), (g) and (h).

(j) Twin primes are two prime numbers whose difference is 2. The numbers 3 and 5 are twin primes, can you find more pairs of twin primes, if so how many?

(k) Consider the expression $n^2 + 3n + 1$. For what values of n do you get prime numbers? Give a value of n for which the above expression is not a prime number.

How to Generate Prime Numbers

Write down all the numbers from 1 to 50. To start with, your sheet will look something like this:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |

Questions:

Using Definition 2, is 1 a prime number? How about 2? Cancel out the number 1. After 2 cancel all the multiples of 2 in your list. What does the list look like now? Except for 2, what type of numbers are you left with?

Is 3 a prime number? Except for 3 cancel all the multiples of 3. How many numbers are you left with? (2, 3, 5, 7, 11, ... 49). (i.e. 18) What is the next prime number? Cancel out multiples of that prime number except the prime itself. How many numbers are you left with? How far do we have to test to find all the prime number up to 50? How many prime numbers do we have from 1 to 50? (Answer: 15)

What do you think is the number of primes between 50 and 100 (same as above, less or more)?

Here is a computer program which will find all the prime numbers less than any number you want

```
  3  INPUT K, N
  5  FOR X = K TO N
 10  LET P = SQR (X)
 20  LET M = INT (P)
 30  FOR Y = 2 TO M
 40  LET Z = X/Y
 50  IF Z = INT (Z) THEN 80
 60  NEXT Y
 70  PRINT X
 80  NEXT X
 90  GO TO 3
100  END
```

Note to the teacher: Give the above program to each student, ask him to run the program for different sets of 100 numbers and count how many prime numbers he gets. Make a table for the number of primes like the following:

| X | No. of Primes | Ratio (No. of Primes/100) |
|---|---|---|
| 0 - 100 | ? | |
| 100 - 200 | | |
| 200 - 300 | | |

Ask the students to draw the graphs between x versus the number of primes. Can you make a conjecture about the number of primes?

## III. Divisors

Definition: B is a divisor of A if the remainder of A divided by B is 0.

Let us suppose A to be 6; the positive divisors of 6 are 1, 2, 3 and 6. In the following investigation we shall only consider the positive divisors.

Part I

(a) Find all the numbers between 1 and 50 with exactly one divisor.

(b) Find all the numbers between 1 and 50 with exactly two divisors.

(c) What generalization is suggested by the numbers which you have found in (b)?

(d) Find all the numbers between 1 and 50 with exactly three divisors.

(e) What generalization is suggested by the numbers which you have found in (d)?

(f) Find all the numbers between 1 and 50 with exactly four divisors.

(g) What generalization is suggested by the numbers which you have found in (f)?

(h) Find all the numbers between 1 and 100 with exactly five divisors.

(i) What generalization is suggested by the numbers which you have found in (h)?

(j) How many divisors are there of the numbers which may be written as $p^a$ where p and a are both prime numbers? Where a prime as a is not prime, make a generalized statement.

(k) How many divisors are there of numbers which may be written as $p_1^a \cdot p_2^b$ where $p_1$ and $p_2$ are distinct primes?

(l) How many divisors are there of numbers which can be written as $p_1^a \cdot p_2^b \cdot p_3^c$ where $p_1$, $p_2$ and $p_3$ are distinct primes?

(m) Make a generalized statement from (h) (l).

(n) Characterize all the numbers with 12 divisors.

Part II

If in the list of divisors of the given number A, you do not include the number itself, then all the remaining positive divisors of the number are called proper divisors.

(a) Find the proper divisors of each of the numbers from 1 to 50.

(b) Make a table as follows:

| no. | Proper Divisors | Sum of the Proper divisors |
|---|---|---|

1) What are the numbers where the sum of proper divisors of each number is less than the number itself? (These numbers are called deficient numbers.)

2) What are the numbers where the sum of proper divisors of the number is equal to the number itself? (These are called perfect numbers.)

3) What are the numbers where the sum of the proper divisors of each number is greater than the number itself? (These are called abundant numbers.)

4) Can you find a deficient number greater than 50?

Short Investigation

The purpose of this investigation is to discover the uniqueness of prime factorization, the Fundamental Theorem of Arithmetic.

(a) How many ways can you write each of the numbers from 2 - 15 as the product of primes? (Note: 2 x 3 and 3 x 2 are considered the same)

(b) What pattern do you see among products of the numbers?

(c) Which of these numbers can you write as the product of primes?

(d) How many ways can you write 48 and 72 as the product of primes?

(e) Make a generalization about the patterns you observe.

1) Your generalization should be something like the following statement:

Every composite number $n$, can be written in the form $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, where $p_i$ is a distinct prime $a_i \geq 1$ and $k \geq 2$.

Short Investigation

The purpose of this investigation is to define the greatest common factor and to determine it using the results of the previous investigations.

(a) Take two numbers, 12 and 18. Find the positive integers which divide both numbers with no remainder.

(b) Which is the greatest of these numbers?

(c) Repeat above steps for 24 and 72.

(d) Give a defination for the greatest common factor of 2 numbers.

Definition: For any two positive integers a and b, an integer c is the greatest common factor if $\frac{a}{c}$ and $\frac{b}{c}$ are integers with no common divisor.

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## IV. A Geometrical Interpretation of Greatest Common Divisor

Part I

There was a time when early mathematicians who were interested in geometry had only the straight edge and compass at their disposal. Suppose that these were the only tools you had.

Question 1: If you were given the two line segments $L_1$ and $L_2$ in Figure 1, could you use a compass and a straight edge to construct the longest line segment whose length divides the lengths of both $L_1$ and $L_2$ an integral number of times.

$L_1$

$L_2$

Question 2: Is it possible to do this for any two line segments, whatever their lengths?

Part II

A Possible Classroom Approach

Give each student a ruler and a compass and hand each student a sheet of paper with four sets of two lines drawn on it of lengths (12, 6). (20, 6), (16.9, 9.1) and (17, 8) units (preferably milimeters). Now ask the students if they can answer question 1 for each set of two lines and if after doing this, they can guess the answer to question 2.

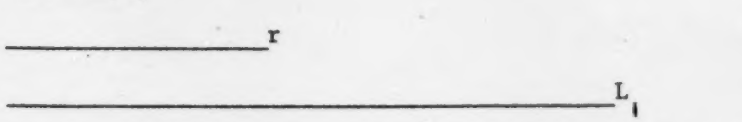
One method for solving the problem using the two lines in figure 1 is as follows:

Step 1: Open the compass equal to the length of $L_1$ and starting at one end of $L_2$, mark off this length onto $L_2$ as many times as you can.

You should find that you can mark off the length of $L_1$ 3 times with some left over. Calling the left-over portion r, we can write:

(a) $L_2 = 3L_1 + r$

Step 2: Now open the compass equal to the length of r and mark off this length onto $L_1$ as many times as you can.




Here we find that the length of r can be marked off onto $L_1$ 2 times with some left over. Calling this leftover portion $r_1$, we write

(b) $L_1 = 2r + r_1$ .

Step 3: Next we repeat this operation, marking off the length of $r_1$ onto r and find that $r_1$ divides r exactly 2 times, i.e.,

(c) $r = 2r_1$ .

From equations (a), (b) and (c), we have

$$L_1 = 5r_1 \text{ and } L_2 = 17r_1 .$$

This tells us that the length of line segment $r_1$ divides the lines $L_1$ and $L_2$ an integral number of times.

Questions

a) Can you find a line segment less than the length of $r_1$ which divides the lines $L_1$ and $L_2$ an integral number of times?

b) Can you find a line segment of length greater than the length of $r_1$ which divides $L_1$ and $L_2$ integrally?

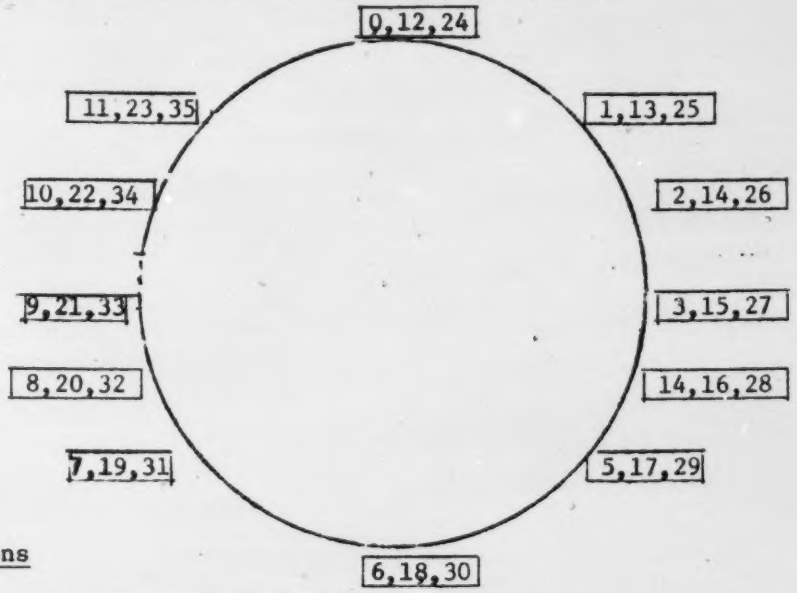
(c) Would you guess that $r_1$ is the longest line which divides $L_1$ and $L_2$ integrally? Can you prove this?

(d) Can you generalize the method of steps 1 to 3 above so as to apply to any two line segments?

(e) Can you give a method for finding the largest common divisor of the positive integers 4 and 14? 12 and 53? How about any two positive integers?

## V. Congruences

Construct a 12 hour clock, using twelve 2" by 1" boxes in which you place the numbers from 0 to 35 as given below.




Questions

(a) What numbers do you have in box 3?

(b) What are the differences between pairs of numbers?

(c) What numbers do you have in box 5?

(d) What are the differences between pairs of numbers?

(e) What numbers do you have in box 12?

(f) What are the differences between pairs of numbers?

(g) What is the relationship between the set of differences in questions (b), (d) and (f) and the number of boxes?

(h) Construct an algebraic equation that will generate the numbers in each box.

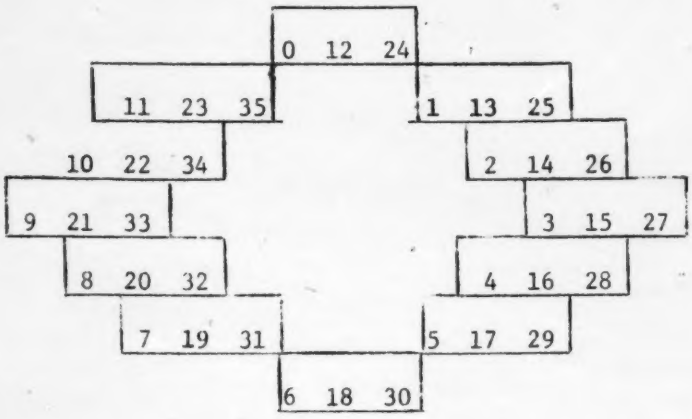
(i) Suppose all positive whole numbers are placed in these boxes, will your algebraic equation generate the numbers in each box? If so, tell in which box you place each of the numbers from 1 to 100.

(j) Construct an 8 hour clock, 6 hour, and a 4 hour clock.

(k) Will your algebraic equation in (h) work for (j)? If not, how can you modify it so that it will?

(l) Can you generalize to make your equation work for the numbers of a given hour clock?

( Note: Any two numbers in the same box are said to be congruent modulo (number of boxes). An example for the 12 hour clock: $1 \equiv 13 \pmod{12}$ )



Solve the following Congruences:

1. $7 \equiv 12 \pmod{m}$
2. $3X \equiv 8 \pmod{7}$ (more than one answer)
3. $7X \equiv 2 \pmod{12}$
4. $6X + 2 \equiv 13 \pmod{7}$
5. $7X + 5 \equiv 2 \pmod{10}$

Exercises - (Leading to divisibility tests)

Find r such that $0 \leq$ 8

1. $10 \equiv r \pmod{9}$, $100 \equiv r \pmod{9}$, $10^3 \equiv r \pmod{9}$ $10^n \equiv r \pmod{9}$ where n is a positive integer.
2. $4 \times 10 \equiv r \pmod{9}$, $7 \times 10^2 \equiv r \pmod{9}$, $9 \times 10^3 \equiv r \pmod{9}$, $2 \times 10^3 \equiv r \pmod{9}$.
3. $a \times 10^n \equiv r \pmod{9}$, n is any positive integer and a is a digit (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)
4. $5 + 7 \times 10 \equiv r \pmod{9}$, $3 + 1 \times 10 \equiv \pmod{9}$, $6 \times 10 + 3 \equiv r \pmod{9}$, $108 = 8 + 1 \times 100 \equiv r \pmod{9}$, $472 = 2 + 7 \times 10 + 4 \times 100 \equiv r \pmod{9}$.
5. What numbers are congruent to zero modulo 9?
6. Can you see a method which will enable you to determine whether or not an integer is divisible by 9 from its decimal (Base 10) representation?

Similarly one can use congruences to determine divisibility tests for 3 and 11.

Definition: The numbers a and b, are congruent modulo n if, and only if, the remainders are equal when a and b are divided by n.

Show that any number is congruent (modulo n) to the remainder in the division of the number by n.

Notice that: $1 + 2 = 3 \equiv 0 \pmod 3$

$4 + 2 = 6 \equiv 0 \pmod 3$ and $4 \equiv 1 \pmod 3$

$7 + 2 = 9 \equiv 0 \pmod 3$ and $7 \equiv 1 \pmod 3$

$103 + 2 = 105 \equiv 0 \pmod 3$ and $103 \equiv 1 \pmod 3$

$1 + 14 = 15 \equiv 0 \pmod 3$ and $14 \equiv 2 \pmod 3$

$1 + 5 = 6 \equiv 0 \pmod 3$ and $5 \equiv 2 \pmod 3$

$1 + 77 = 78 \equiv 0 \pmod 3$ and $77 \equiv 2 \pmod 3$

$145 + 65 = 210 \equiv 0 \pmod 3$ and $145 \equiv 1 \pmod 3$

This indicates that if $a \equiv c \pmod n$ and $b \equiv d \pmod n$ then $a + b \equiv c + d \pmod n$. Show, similarly that if $a \equiv c \pmod n$ and $b \equiv d \pmod n$ then $a \times c \equiv b \times d \pmod n$.

Point out that the modular must be the same in all parts for these rules to be true, e.g., $5 \equiv 8 \pmod 3$ and $2 \equiv 10 \pmod 4$ but $5 + 2 \not\equiv 8 + 10 \pmod 3$ or $\pmod 4$

## VI. Guess My Number

"Guess My Number" is a game which is useful for reviewing this unit. The rules are as follows:

1) There are two teams. Each team has a captain who is the only member of the team authorized to ask queations of the other team or answer the other team's questions.

   [Other possibilities: (a) captainship rotates with each change of turn.

   (b) captainship rotates with each question and answer.]

2) One tean (A) chooses a positive integer no greater than 100. The other team (B) tries to guess this number. The number is written, in advance, on a piece of paper.

3) Team B may make <u>one</u> **free** guess each question.

   [Alternatively, each guess of a number counts as a question.]

4) The only permissible answers to questions are "yes" and "no". Hence all questions must be worked accordingly.

5) The same form of a question may not be used twice by the same team in a turn.

If the question "Is the number divisible by 3?" is asked then no more questions using "divisible" may be asked. Words such as "multiple," "divisor" and "to divide" may be acceptable.

6) When the number is correctly guessed, team B is awarded a score of 10 points less one point for each question asked. Negative scores are possible.

7) The teams alternate between choosing a number and guessing the other team's number. The game ends when one team has scored 50 points or more and each team has had an equal opportunity to score.

8) If an incorrect answer is given to a question and not changed before the guessing team makes a guess or asks another question, then the guessing team is awarded a full score of 10 points.

# VII. Pythagorean Systems

## A. Pythagorean Triples

Choose two positive integers u anf v such that one is even while the other is odd. Let $v > u$ and the greatest common factor of u and v be 1.

Let $a = v^2 - u^2$,

$b = 2uv$,

$c = u^2 + v^2$ .

1. Choose several values of u and v satisfying the above conditions. What relationship exists among a, b and c? How does this information relate to triangles?

2. Make a table of several values of a, b and c and make as many observations as possible.

It turns out, of course, that a, b and c are so related such that $a^2 + b^2 = c^2$; which is an algebraic interpretation of the Pythagorean Theorem on right triangles. For example, if u = 1 and v = 2 we get

$a = 2^2 - 1^2 = 3$

$b = 2\ (1)\ (2) = 4$

$c = 1^2 + 2^2 = 5.$

Hence: $3^2 + 4^2 = 5^2$, which represents the measures for a right triangle.

These integers, called Pythagorean triples, are used in Diophantine equations of the second degree. Diophantine equations are so named because they were first studied by the Greek mathematician, Diophantus, who lived in the third century. In general, the Diophantine equations are of the form $a^n + b^n = c^n$, where the solutions are required to be integers.

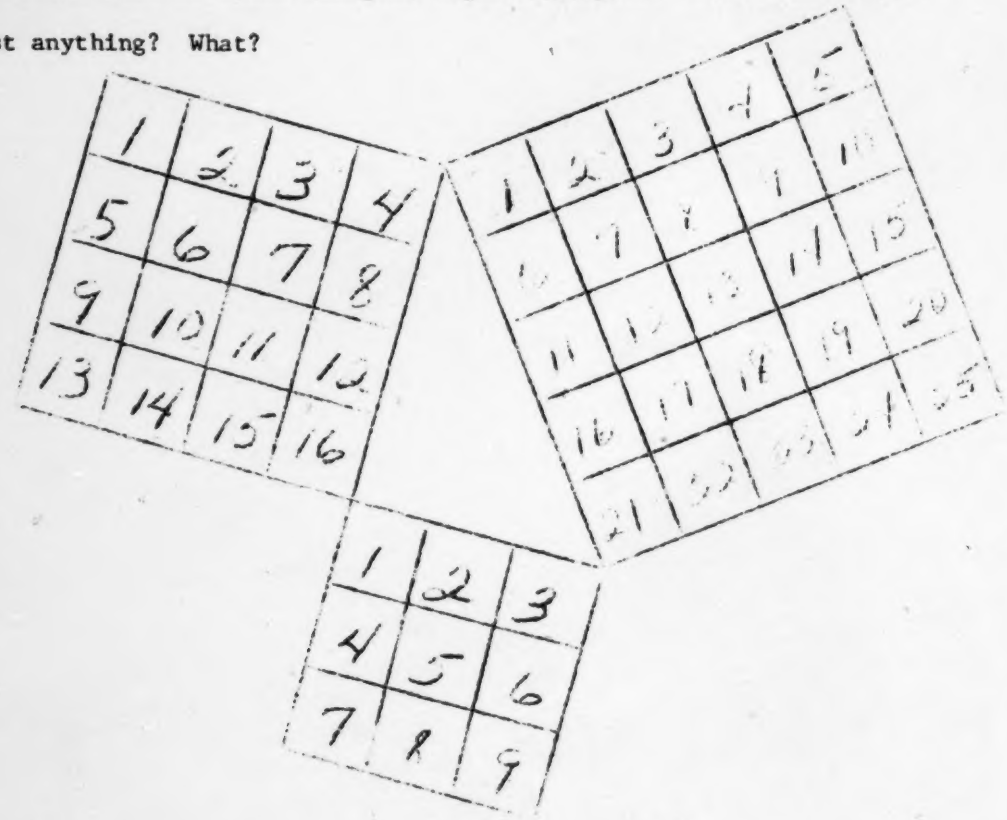
The following table contains examples of Pythagorean triples.

| u | v | a | b | c |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 2 | 3 | 5 | 12 | 13 |
| 3 | 4 | 7 | 24 | 25 |

| u | v | a | b | c |
|---|---|---|---|---|
| 1 | 4 | 15 | 8 | 17 |
| 4 | 5 | 9 | 40 | 41 |
| 2 | 5 | 21 | 20 | 29 |
| 5 | 6 | 11 | 60 | 61 |
| 1 | 6 | 35 | 12 | 37 |
| 6 | 7 | 13 | 84 | 85 |
| 4 | 7 | 33 | 56 | 65 |
| 2 | 7 | 45 | 28 | 53 |
| 7 | 8 | 25 | 112 | 113 |
| 5 | 8 | 39 | 80 | 89 |

1. You may notice that in each triple, at least one of the numbers is divisible by 3, 4, or 5. For example in the triple 5, 12, 13, 12 is divisible by both 3 and 4; and, of course, 5 is divisible by 5. In the triple: 8, 15, 112, 8 and 112 are divisible by 4; 15 is divisible by 3 and 5. Is this true in general?
2. You will notice that in some cases the triples contain pairs of consecutive numbers. For example: 12 and 13 in (5, 12, 13), and 6 and 7 in (6, 7, 13). Can you comment, in general, when this will be true?

What interested the ancient scientists most, were the special cases of right triangles where the three sides were all integers. Can you discover some of these integral-right triangles? Does this diagram suggest anything? What?



In ancient times, in Babylonia probably, 1,500 years before Christ, men built large fortifications and made their walls perfectly erect and their corners perfectly perpendicular. Archeologists have discovered that they used, for this purpose, a rope that had knots tied in it at regular intervals. Can you, looking at the triangle with sides 3 and 4 and hypotenuse 5, imagine how this knotted rope helped those men to make their perpendicular constructions?

Look up proofs of the Pythagorean Theorem in different books, particularly old textbooks from 50 years ago or so, and see if you can understand their proofs. Which proofs do you like best? Why?

Because of the Pythagorean Theorem, the geometric problem of finding right triangles with integer lengths for the three sides, is reduced to the numerical problem of searching for three integers (a triple of integers) so that the sum of the squares of the first two equals the square of the third. That is, if you can find two perfect squares that add up to another perfect square, then you have found a "Pythagorean triple" so called. Can you systematically discover all the Pythagorean triples there are which use integers from one to two hundred? How many do you expect you will find? See what happens.

If you have a computer handy, that uses the BASIC language, you might try this program:

```
10 FOR X = 1 TO 200
20 FOR Y = (X + 1) TO 200
30 LET Z = SQR (X↑2 + Y↑2)
40 LET W = Z - INT (Z)
50 IF W > 0 THEN 70
60 PRINT X, Y, Z
70 NEXT Y
80 NEXT X
90 END
```

What do you think it will do? What will it print out? Check it out and see.

Questions about triples (X,Y,Z) of whole numbers satisfying the restriction that $X^2 + Y^2 = Z^2$, i.e. the Pythagorean triples.

1. Some of the triples seem to be "replicas" of others. Does that seem to be the case to you? How would you define "replica" in that case?
2. Suppose the "replicas" were tossed out of a list (say the list we got from the computer), then are there any interesting things in the list that remains? Any interesting properties of the triples? What about the questions of even-odd-ness of the numbers?
3. Does each number appear in one of the spaces of some triple on the list? If so, why? If not, why not?

4) Can you find triples where two numbers repeat? If not, why not?

5) Can you find triples where two of the numbers are consecutive? Any pattern there? How many such numbers are there?

6) What about triples where two of the numbers differ by two? or three?. etc.

7) Do you think there is an EASIER way to write out all the Pythagorean triples? Do you see any pattern to all the triples in this list?

The question of two squares adding up to a third square suggests taking a look at the numbers that are equal to the sum of two squares:

| | 1 | 4 | 9 | 16 | 25 | 36 | 49 |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 5 | 10 | 17 | 26 | 37 | 50 |
| 4 | - | 8 | 13 | 20 | 29 | 40 | 53 |
| 9 | - | - | 18 | 25 | 34 | 45 | 58 |
| 16 | - | - | - | 32 | 41 | 52 | 65 |
| 25 | - | - | - | - | 50 | 61 | 74 |
| 36 | - | - | - | - | - | 72 | 85 |
| 49 | - | - | - | - | - | - | 98 |

Here are some of them: 2, 5, 8, 10, 13, 17, 20, 25, 26, 32, 34, 37, ... . Do these numbers get bigger and bigger? Do they fill in

the number sequence more and more? Let us call these numbers "double-squares" just to be able to talk about them. The question is, then, as you look at higher and higher numbers, are more of them "double-squares"? Or do "double-squares" numbers peter out? Become less and less frequent?

What happens if you multiply two double-square numbers. Do you get another double-square number?

What happens if you multiply a double-square number with one that is not?

Are there "special" double-square numbers? Are there any of them that are more interesting than the others?

What about numbers that are the sum of three squares? Are all numbers the sum of three squares? Suppose you multiplied two triple-squared numbers?

What about the numbers that are the sum of four squares? Is every number representable as the sum of four squares? Five squares? Six?

What about cubes? Is every number representable as the sum of three (or four, or n) cubes? How many?

Questions

1. Consider the triples (3, 4, 5) (6, 8, 10).
   Is it fair to speak of these as separate triples? Or are they the same?

   Suppose that you had a triple (x,y,z) and you multiplied each of the three numbers by some other number, t, is the new set of three (tx, ty, tz) a Pythagorean triple? Is it necessary to keep those extra triples around?

2. You may have noticed that, when the "extra" triples or the "replicas" are thrown out of the list, the remaining triples have the following peculiar properties: first, the third number in each triple (the hypotenuse) is always an odd number; and second, of the first two numbers, one is odd and the other is always even. Why is this the case?

   If you write down all the possibilities for the even-ness and the odd-ness of the three numbers in a triple, making no assumptions or restrictions on what a number might be, how many combinations are there? i.e. (even, even, even), (even, even, odd), etc. Now, go back and take a look at each of these combinations, each of these prospective triples. How about the combination (even, even, even)?

Is that going to be on the list at this point? (Remember, we have tossed out of the list all the triples in which there are factors common to all three numbers in the triple; (6, 8, 10) has been tossed out). What about the triple (odd, odd, odd)? Could that be possible? Are there any examples in the list like that? Why not? Try to construct an example like that and see why it fails every time.

It might be useful to consider the following proposition:

Proposition: If X is the square of some integer, then X is either equal to a multiple of four, or one greater than a multiple of four.

Every number can be written as a multiple of four plus 1, 2, 3 or 0. (Think about that.) It might then be surprising that squares take on only the two forms: multiples of four plus either zero or one, never two or three. Why is that? (Do cubes work out the same way?)

Proof: Assume X is even, therefore $X = 2N$ for some integer N.
Then $X^2 = 4N^2 = 4K$, where K is some integer.
Suppose, now, that X is odd; it can be written as
$X = 2M + 1$, for some M.
Then, $X^2 = 4M^2 + 4M + 1 = 4 (M^2 + M) + 1 = 4L + 1$,
for some integer L.

Suppose you looked now at the possibility of a triple (odd, odd, Z). Could that happen? If the first two terms were odd, then their squares will be of the form $4N + 1$, and $4M + 1$, for some integers N and M. But the sum of the squares must be equal to the square of the Z, whatever it is. This sum will be of the form $4(N + M) + 2 = 4L + 2$, which cannot equal the square of either an even or an odd number. Thus, the first two numbers in a triple cannot both be odd.

What possibilities does that leave?

3. Can one of the numbers in a triple be one? Which do not show up in our list? Why not? Suppose we had a triple (2, Y, Z). Could that not be satisfied with some Y and Z? Why not?

4. Suppose you had (X, X, Z). What rule would that violate? Why do we not get triples like that? What about (X, Y, X)?

5. Look at the triples where there are consecutive integers: (3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61), (13, 84, 85), etc. Do you notice any pattern?

   Is it useful to consider the following identity:

$$(2N + 1)^2 + (2N\,(N + 1))^2 = [2N\,(N + 1) + 1]^2\ ?$$

6. Consider the triples where the numbers differ by 2: (6, 8, 10); (8, 15, 17); (10, 24, 26); (12, 35, 37); (14, 48, 50); etc. What pattern do you see here? Since all the first numbers are increasing by two each time, this suggests we seek to begin the series with a triple like (4, ___, ___). Or even before that? (2, ___, ___)?

   Look at the last pair of numbers in each triple: 8, 10; 15, 17; 24, 26; 35, 37; 48, 50; etc. Does that suggest a pattern?

   Might it be useful to look at the following identity: $(2N)^2 + (N^2 - 1)^2 = (N^2 + 1)^2$ ?

7. This part will be devoted to showing that one can devise a set of formulas for (X, Y, Z) so that one can crank out the relatively prime triples satisfying the condition $X^2 + Y^2 = Z^2$, that is, the Pythagorean, relatively prime, triples. (Reference: Notes to Chapter 13 of Famous Problems of Mathematics, by Heinrich Tietze, Graylock Press, New York, 1965, $10.00. It is full of interesting questions that allow the student to do things right away.)

We have already seen that if (X, Y, Z) is a relatively prime Pythagorean triple , then one of the first two numbers is even, the other odd, and the last number odd as well. Let us assume, since there is no way to distinguish X from Y - both sides of the right triangle - that X is odd and Y is even. Rewriting the relation $Y^2 = Z^2 - X^2 = (Z - X)(Z + X)$; and since both Z and X are odd, then (Z - X) and (Z + X) are both even. Why?

So we may write them as (Z-X)= 2R and (Z+X)=2S. Here S > R. Adding and subtracting these two equations, we get: Z = R + S, and X = S - R. Show that S and R have no common factor except 1. (If they did what would you be able to say about common factors of Z and X, and then Y? So, both R and S cannot be even. But, they cannot both be odd either. (If they were, what would that say about Z and X?) So one is odd and the other is even.)

Substituting the expressions for Z and X in terms of R and S into $Y^2 = (F - X)(F + X)$ we get $Y^2 = 4$ RS. So Y is even. But then, every even number squared equals an expression of the form $W^2$. (See Proposition page 38.) So, $W^2 = RS$. Since R and S have no common factors, and since RS is equal to a square, then every factor that R does have must be repeated, so that R itself is a square, similarly with S.

(Prove that proposition.) Thus, $R = A^2$, and $S = B^2$, for some numbers A and B. Since $S > R$, therefore, $B > A$. A and B are relatively prime, with no common factors. (If they did have a common factor, what would that say about R and S?) So what do we have:

$$X = B^2 - A^2; \quad Y = 2AB; \quad Z = B^2 + A^2$$

Using these formulas, where A and B are relatively prime, and where $B > A$, what can you generate for the triple (X, Y, Z)?

To show that the triple (X, Y, Z) generated by the numbers A and B is a Pythagorean triple, one must show that, no matter what A and B are, the X, Y and Z generated by them satisfies $Z^2 = X^2 + Y^2$. But even if one does that, what has been shown above is a much stronger result. This result is that not only do we get Pythagorean triples by substituting relatively prime values for A and B, but we get every Pythagorean triple that way. After all, we started with any Pythagorean triple (relatively prime), and showed how it could have been generated with A and B by those formulas; it could have been any Pythagorean triple (X, Y, Z) in the previous formulas generating triples. (See comments on numbers 5 and 6, pages 39 and 40). Here, we get all the Pythagorean triples; before we got only some of the triples.

Comments on double-squared numbers, etc., pages 39 and 40.

Suppose a numbaer is a double-squared numbet : It equals $(a^2 + b^2)$ for some integers a and b. Suppose a second number is also a double-squared number: it might equal $(c^2 + d^2)$. Now suppose you multiplied them together. $(a^2 + b^2) \cdot (c^2 + d^2) = ?$

Check out the following expression and see what it might be able to tell you about the product of two double-squared numbers:

$$(ac + bd)^2 + (ad - bc)^2 = ?$$

One notices that the product of two double-squared numbers with each other is also a double-squared number and also that the product of a double-squared number with a square is again a double squared number. The interesting property to notice is that the product of a double-squared numbers (excluding the kind like $a^2 + b^2$) is not only another double-squared number, but it will always be one of the double-squared numbers that appears <u>twice</u>; that is, such a number appears as the sum of two squares, <u>in two</u> ways. Maybe the above identity will help you see how this can happen. Let us look at the identity with the letters replaced by numbers: $a = 1$, $b = 2$, $c = 1$, $d = 3$

$$(a^2 + b^2) = 5,\ (c^2 + d^2) = 10,\ (a^2 + b^2)\ (c^2 + d^2) = 50.$$

Now: $(ac + bd)^2 + (ad - bc)^2 = (3 + 2)^2 + (1 - 6)^2 = 5^2 + 5^2 = 50.$

But c and d could be switched around, in this formula, to get another way of looking at 50 as the sum of two squares. Let $c = 3$, and $d = 1$. $(c^2 + d^2)$ is still 10 . But $(ac + bd)^2 = (3 + 2)^2 + (1 - 6)^2 = 5^2 + 5^2 = 50.$

Try other combinations for a, b, c, and d and see what you get. What happens if the first double-squared number is something like $(3^2 + 3^2)$; can the product of this number and another double-square number equal a number that can be written as the sum of two squares (a double-square) in two different ways?

A little hand in calculating suggests very strongly that every integer is the sum of, at most, four squares. Try it and see. Do you think you can find a number which is not?

To find a mathematical proof of this conjecture, look at Chapter 9 of The Enjoyment of Mathematics by Hans Rademacher and Otto Torplitz, a Princeton paperback published in 1966.

C. On Ledet Numbers

Consider the set of numbers 3, 5, 7. Are these Pythagorean numbers? (Do they satisfy the relation $c^2 = a^2 + b^2$?) Can you find a relation similar to the Pythagorean relation which is satisfied by these numbers?

One such relation is $c^2 = a^2 + b^2 + ab$. Triples of positive integers which satisfy this relation are called Ledet triples or Ledet numbers. (See the Similarity and Trigonometry unit written at the Thirteen College Curriculum Program Summer Conference.) Can you find any other Ledet numbers?

Examine the following computer program written in the Basic language:

```
 10 FOR B = 1 TO 100
 20 FOR A = 1 TO B
 30 FOR C = 1 TO A + B - 1
 40 IF C SQR(A 2 + B) THEN 100
 50 LET D = (C 2 - A  2 - B  2)/(A*B)
 60 IF D 1 TEEN 100
 70 PRINT A:B:C
100 NEXT C
110 NEXT A
120 NEXT B
130 END
```

D. Ledetian Triples

For each pair of positive integers u, r, let

$a = r(2u + 1)$

$b = r(3u^2 + 2u)$

$c = r(3u^2 + 3u + 1)$

1. Choose several values of u and r, and for each pair of values, find the corresponding values for a, b and c. What relationship exists among a, b and c? How does this information relate to Ledetian triangles?

2. Make a table of several values of a, b and c and make as many observations as possible. It turns out, of course, that a, b and c are related such that $a^2 + b^2 + ab = c^2$, which is an algebraic interpretation of the Ledetian Theorem on 120° triangles. For example, if u = 1 and v = 1, we get:

$a = 1(2 \cdot 1 + 1) = 3$

$b = 1(3 \cdot 1^2 + 2 \cdot 1) = 5$

$c = 1(3 \cdot 1^2 + 3 \cdot 1 + 1) = 7$ .

Hence, $3^2 + 5^2 + 3 \times 5 = 7^2$, which represents the measures for a 120° triangle.

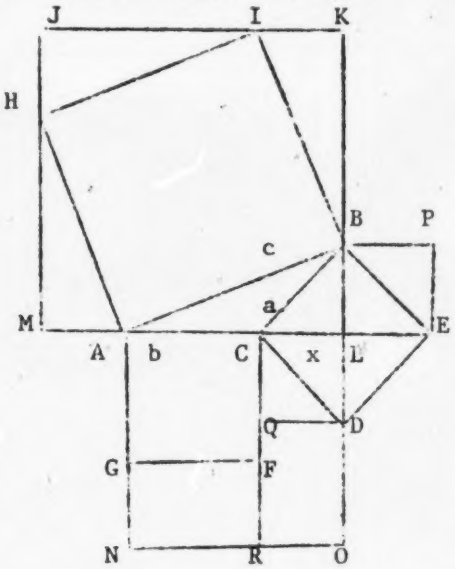
These integers called Ledetian Triples, are used in the above mentioned diophantine equations of the second degree. The equations called the Ledetian equations are so named because they were first studied by the contemporary American mathematician Nancy Ledet. The following table contains examples of Ledetian triples.

| u | v | a $(2n + 1)$ | b $(3n^2 + 2n)$ | | $(3n^2 + 2n) + n + 1)$ |
|---|---|---|---|---|---|
| 1 | 1 | 3 | 5 | | 7 |
| 1 | 2 | 6 | 10 | | 14 |
| 2 | 1 | 5 | 16 | (n+n) | 19 |
| 2 | 2 | 10 | 32 | | 38 |
| 3 | 1 | 7 | 33 | | 37 |
| 4 | 1 | 9 | 56 | | 61 |
| 5 | 1 | 11 | 85 | | 91 |
| 6 | 1 | 13 | 120 | | 127 |
| | | $(2n + 1)$ | $(3n^2 + 2n)$ | | $(3n^2 + 3n + 1)$ |
| 7 | 1 | 15 | 161 | | 169 |
| 8 | 1 | 17 | 208 | | 217 |
| 9 | 1 | 19 | 261 | | 271 |
| 10 | 1 | 21 | 320 | | 331 |
| | | $(v(2u + 1))$ | $(v(3u^2 + 2u))$ | | $v(3u^2 + 3u + 1)$ |

1. You may notice that in each triple at least one of the numbers is divisible by 3, 5 or 7. For example, in the triple (9, 56, 61), 9 is divisible by 3 and 56 is divisible by 7. In the triple (21, 320, 331), 21 is divisible by both 3 and 7 and, of course 320 is divisible by 5. Is this true in general?

2. You may notice that in no case does a triple contain a pair of consecutive numbers. In general, can you determine if this is true?

What contemporary mathematicians are interested in are special cases of 120° triangles where the three sides are all integers. Can you discover some of those integral 120° triangles which are not given by the previous formulas? Does this diagram suggest anything? What?



Questions About Triples (X, Y, Z) of Whole Numbers Satisfying the Restriciton that $X^2 + Y^2 + XY = Z^2$, i.e., the Ledetian Triples

1. Some of the triples are "replicas" of others. How do you define "replica" in the case of Ledetian Triples?
2. Suppose the "replicas" were tossed out of the computer list. Are there any interesting properties in the list that remain? What about questions of even-oddness of the numbers?
3. Does each number appear in one of the spaces of some triple on the list? If so, why? If not, why not?
4. Can you find triples where two numbers are repeated? If not, why not?
5. Can you find triples where two of the numbers are consecutive? Where two of the numbers differ by 2? Any pattern there? How many such numbers are there?
6. Do you think there is an easier way to list the Ledetian Triple? Do you see any pattern in the list on page b?

The question of the squares of two numbers and their product adding up to a square suggests that we take a look at a table of these sums . . .

| a, | b<br>$a^2$ $b^2$ | 1<br>1 | 2<br>4 | 3<br>9 | 4<br>16 | 5<br>25 | 6<br>36 | 7<br>49 |
|---|---|---|---|---|---|---|---|---|
| 1, | 1 | 3 | 7 | 13 | 21 | 31 | 43 | 57 |
| 2, | 4 | - | 10 | 19 | 28 | 39 | 52 | 67 |
| 3, | 9 | - | - | 24 | 37 | 49 | 63 | 79 |
| 4, | 16 | - | - | - | | | | |
| 5, | 25 | - | - | - | - | | | |
| 6, | 36 | | | | | | | |
| 7, | 49 | | | | | | | |

Here are some of them: 3, 7, 10, 13, 19, 24, . . . Do these numbers continue to fill in the number sequence? Let us call these numbers "Ledetian sums" for convenience. What happens when you multiply two "Ledetian sums"? Do you get another "Ledetian sum"?

What happens when you multiply a "Ledetian sum" number with one that is not a "Ledetian sum" number? Are there "Special" "Ledetian sum" numbers? Are there any that are more interesting than others?

Questions

1. Consider the triples (3, 5, 7) and (6, 10, 14). Is it useful to speak of these as separate triples?

Suppose you had a triple (x, y, z) and you multiplied each of the three numbers by some other number, t. Is the new set of three numbers (tx, ty, tz) a Ledetian triple? Is it necessary to keep those extra triples around?

2. You may have noticed that when the "extra" triples or the "replicas" are thrown out, the remaining triples have the following properties:

a. The first and third numbers of each triple are always odd,

b. The second number of each triple may be odd or even. Why is this so?

c. The difference between c and b is $\frac{a+1}{a^2}$. Is this true for all Ledetian numbers?

Now, consider the collections of all relatively prime Ledetian triples (a, b, c), where $c^2 = a^2 + b^2 + ab$.

a. If both a and b are odd, then c is odd. Why?

b. If one of either a or b is odd and the other even, then c is odd.

3. Since c is even, if,and only if, both a and b are even, then c is always odd for this collection. Why?

Hence, the largest number in a relatively prime Ledetian triple is always odd. What possibilities does that leave?

4. Can one of the numbers in a triple be one? Suppose we had a triple (2, y, z). Could that not be possible to satisfy for some y and z? Why not?

5. Suppose you had (x, x, z). What rule would that violate? What about (x, y, x)?

6. Using the relatively prime triples from the table on page b, under what conditions is b even? (Look at how the evenness of b relate to a).

7. Look at the following triples: (3, 5, 7),(7, 8, 13),(9, 15, 21), (11, 24, 31) and (13, 35, 43).

Do you notice any pattern? In each case, determine the sum of the two larger numbers. How does this sum relate to the first number?

Is it useful to consider the following identity:

$(2N + 1)^2 + (N^2 - 1)^2 + (2N + 1)\ (N^2 - 1) = (N^2 + N + 1)^2$ ?

Exercises

1. A builder of prefabricated roof trusses found that he could satisfy the demands of most styles of architecture if he set up his jig to form 120° angles. How would you use knotted ropes to accurately design a 120° roof?

## VIII. Linear Diophantine Equations

On home coming day the A.K.A. sorority gave a dance for the home coming alumni. The tickets for men were 3 dollars each and tickets for women were 2 dollars each. They collected 30 dollars. How many men and women were there?

To put the above problem in algebraic symbols it will read:

$3x + 2y = 30$

where x is the number of men and

y is the number of women.

as (x, y) both have to be integers. Under this restriction,

a) How many solutions do you think the above equation will have?

b) Suppose we remove the restriction that (x, y) are integers, how many solutions will it have?

c) Can you write the equation in mod 2? mod 3 form?

d) Can you write all your solutions in the form:

$$x = r + bt$$
$$y = s - at$$

where $b = 2$, $a = 3$

and t is an integer what are the values of (r, s)?

The equations in which both (x, y) are whole numbers are named the greek mathematician Diophantus who lived in the third century and are called Diophantine equations. A Diophantine equation may have no solutions, may have a finite number of solutions or may have an infinite number of solutions.

For the general equation $ax + by = c$, if $x = r$ and $y = s$ is a particular solution, then all the solutions can be determined by the relation: $x = r + bt$

$$y = s - at$$

where t may be a positive or negative whole number.

When $t = 0$, we have $x = r$, $y = s$, a particular solution. A Diophantine equation of the first degree maybe considered as a congruence relation of the first degree. So, for the general equation

$ax + by = c$ may be considered as

$ax \equiv c \pmod{b}$

$by \equiv c \pmod{a}$.

Now, it is not surprising that $ax \equiv c$ may have no solution. For the equation to have a solution any common factor of (a, b) should divide c.

Problem

Discuss which of the following equations have integral solutions.

a) $13x + 55y = 1$

b) $7x + 21y = 14$

c) $7x + 13y = 1$

d) $3x + 2y = 5$

e) $2x + 7y = 15$

f) $x + 6y = 15$

The Problem of Determining a Particular Solution

illustrate how to solve a given equation. Though this method seems long, it is easily understood.

Suppose we want to solve the equation:

$$7x + 13y = 1$$

Here $7x = 1 - 13y$

or $x = \frac{1 - 13y}{7}$

As x has to be an integer, we can say:

Let $x = \frac{1 - 13y}{7} = t$ where t is an integer

$\therefore x = t \quad (1)$

$1 - 13y = 7t$

or $13y = 1 - 7t \quad (2)$

Let us try to find some values of t which will solve equation number 2,

$t = 0$, (Does it yield y to be an integer?)

Let $t = 1$ (Does it yield y to be an integer?)

Let $t = 2$ (Does it yield y to be an integer?)

Yes, so $y = -1$ $x = 2$

The first solution is $x = 2$, $y = -1$

Now, we can write all the solutions.

$$x = r + bt$$

$$y = s - at$$

$$x = 2 + 13t$$

$$y = -1 - 7t$$

Question

Find some more solutions and test whether they satisfy the equation:

$$7x + 13y = 1$$

Using this method, solve the equations in the given problem which you think are soluable.